Applied and Numerical Harmonic Analysis

$$\hat{f}(\gamma) = \int f(x) e^{-2\pi i x \gamma} dx$$

Akram Aldroubi, Carlos Cabrelli
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Editors

New Trends in Applied Harmonic Analysis

Sparse Representations, Compressed Sensing, and Multifractal Analysis

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ANHA Series Preface

The *Applied and Numerical Harmonic Analysis (ANHA)* book series aims to provide the engineering, mathematical, and scientific communities with significant developments in harmonic analysis, ranging from abstract harmonic analysis to basic applications. The title of the series reflects the importance of applications and numerical implementation, but richness and relevance of applications and implementation depend fundamentally on the structure and depth of theoretical underpinnings. Thus, from our point of view, the interleaving of theory and applications and their creative symbiotic evolution is axiomatic.

Harmonic analysis is a wellspring of ideas and applicability that has flourished, developed, and deepened over time within many disciplines and by means of creative cross-fertilization with diverse areas. The intricate and fundamental relationship between harmonic analysis and fields such as signal processing, partial differential equations (PDEs), and image processing is reflected in our state-of-the-art *ANHA* series.

Our vision of modern harmonic analysis includes mathematical areas such as wavelet theory, Banach algebras, classical Fourier analysis, time-frequency analysis, and fractal geometry, as well as the diverse topics that impinge on them.

For example, wavelet theory can be considered an appropriate tool to deal with some basic problems in digital signal processing, speech and image processing, geophysics, pattern recognition, biomedical engineering, and turbulence. These areas implement the latest technology from sampling methods on surfaces to fast algorithms and computer vision methods. The underlying mathematics of wavelet theory depends not only on classical Fourier analysis but also on ideas from abstract harmonic analysis, including von Neumann algebras and the affine group. This leads to a study of the Heisenberg group and its relationship to Gabor systems and of the metaplectic group for a meaningful interaction of signal decomposition methods.

The unifying influence of wavelet theory in the aforementioned topics illustrates the justification for providing a means for centralizing and disseminating information from the broader, but still focused, area of harmonic analysis. This will be a key role of *ANHA. We* intend to publish with the scope and interaction that such a host of issues demands.

Along with our commitment to publish mathematically significant works at the frontiers of harmonic analysis, we have a comparably strong commitment to publish major advances in applicable topics such as the following, where harmonic analysis plays a substantial role:

Biomathematics, bioengineering, and biomedical signal processing;
Communications and RADAR;
Compressive sensing (sampling) and sparse representations;
Data science, data mining, and dimension reduction;
Fast algorithms;
Frame theory and noise reduction;
Image processing and super-resolution;
Machine learning;
Phaseless reconstruction;
Quantum informatics;
Remote sensing;
Sampling theory;
Spectral estimation;
Time-frequency and Time-scale analysis—Gabor theory and Wavelet theory

The above point of view for the *ANHA* book series is inspired by the history of Fourier analysis itself, whose tentacles reach into so many fields.

In the last two centuries Fourier analysis has had a major impact on the development of mathematics, on the understanding of many engineering and scientific phenomena, and on the solution of some of the most important problems in mathematics and the sciences. Historically, Fourier series were developed in the analysis of some of the classical PDEs of mathematical physics; these series were used to solve such equations. In order to understand Fourier series and the kinds of solutions they could represent, some of the most basic notions of analysis were defined, e.g., the concept of "function." Since the coefficients of Fourier series are integrals, it is no surprise that Riemann integrals were conceived to deal with uniqueness properties of trigonometric series. Cantor's set theory was also developed because of such uniqueness questions.

A basic problem in Fourier analysis is to show how complicated phenomena, such as sound waves, can be described in terms of elementary harmonics. There are two aspects of this problem: first, to find, or even define properly, the harmonics or spectrum of a given phenomenon, e.g., the spectroscopy problem in optics; second, to determine which phenomena can be constructed from given classes of harmonics, as done, for example, by the mechanical synthesizers in tidal analysis.

Fourier analysis is also the natural setting for many other problems in engineering, mathematics, and sciences. For example, Wiener's Tauberian theorem in Fourier analysis not only characterizes the behavior of the prime numbers but also provides the proper notion of spectrum for phenomena such as white light; this latter process leads to the Fourier analysis associated with correlation functions in filtering and prediction problems, and these problems, in turn, deal naturally with Hardy spaces in the theory of complex variables.

Nowadays, some of the theory of PDEs has given way to the study of Fourier integral operators. Problems in antenna theory are studied in terms of unimodular trigonometric polynomials. Applications of Fourier analysis abound in signal processing, whether with the fast Fourier transform (FFT), or filter design, or the adaptive modeling inherent in time-frequency-scale methods such as wavelet theory.

The coherent states of mathematical physics are translated and modulated Fourier transforms, and these are used, in conjunction with the uncertainty principle, for dealing with signal reconstruction in communications theory. We are back to the raison d'être of the *ANHA* series!

College Park, MD, USA John J. Benedetto

Foreword

The CIMPA13 Conference which took place in August 5–16, 2013, in Mar de Plata, Argentina, was entitled **New Trends in Applied Harmonic Analysis Sparse Representations, Compressed Sensing and Multifractal Analysis**. The event took place in a friendly atmosphere, encouraging interaction between speakers and participants, among them PhD students, postdocs, and senior scientists. Unfortunately not all the main speakers have been able to provide a written version of their presentation, but in many cases one may find slides of more formal talks through the Internet. General information about the conference can be found at

`http://www.nuhag.eu/cimpa13`

The topics of the articles which appear in this volume reflect the diversity of recent developments in harmonic analysis, both at the level of pure mathematics and applications. Some contributions concern interesting mathematical questions arising from a systematic investigation of structures which have not been sufficiently well explored so far, and others – such as sparsity with respect to non-orthogonal systems – are part of a current trend, related to compressed sensing.

To be more precise, let us take a look at the individual contributions: The first three chapters describe problems related to multifractal analysis (Kathryn E. Hare, Stephane Seuret, and Yanick Heurteaux).

We then find two chapters thematizing the sparsity of wavelet coefficients. In the first contribution (by Vladimir Temlyakov), Lebesgue-type inequalities for greedy approximations are discussed, demonstrating that many of the well-known expansions have the following nice property: Given the set of, say, wavelet coefficients of a given function in some Besov space (because these spaces can be characterized by weighted summability conditions with respect to a given wavelet system), it is a good strategy (not only in the Hilbert spaces setting) to just take more and more of the "large coefficients" in order to approximate the function, in fact with an optimal rate.

In the second chapter in this direction, written by Eugenio Hernandez and María de Natividade, we learn some *results on nonlinear approximation for wavelet bases in weighted function spaces*. Here Bernstein- and Jackson-type theorems for weighted L^p-spaces are provided, showing that wavelet expansions are doing a good job for the approximation of functions in this setting.

The chapter provided by Pete Casazza and Janet C. Tremain discusses *the consequences of the Marcus/Spielman/Srivasta solution to the Kadison-Singer problem* in the context of frame theory with some first glimpse on the consequences within harmonic analysis.

The chapter "Model Sets and New Versions of Shannon's Sampling Theorem" by Basarab Matei presents some interesting insight on universal sampling sets, the so-called model sets and their relations to quasicrystals. While the classical Shannon theorem describes how one can recover a band-limited signal, given the *spectral support* Ω (the support of $\hat{f}$), with a formula which obviously depends on the choice of this set, the new approach discusses situations where the same sampling set can be used (with a more complicated recovery algorithm) for a large variety of sets Ω, as long as their measure is not too big.

The section written by Xianfeng Hu, Yang Wang, and Qiang Wu treats a somewhat unusual and therefore very interesting topic: *Stylometry and Mathematical Study of Authorship*.

The final contribution, entitled "Thoughts on Numerical and Conceptual Harmonic Analysis," provided by the author of this introduction gives a glimpse on a problem within the community of harmonic analysts which should be given a bit more attention: the interaction between principles of abstract (or as he proposes conceptual harmonic analysis) and those who are involved in numerical resp. computational harmonic analysis. While the first group is searching for general structures, the second one is looking for efficient algorithms and their implementation, often using FFT-based algorithms. The aspect lost in this separation of duties is the connection between the two approaches, the question, which function spaces are suitable to describe the errors made by moving from the continuous, to the discrete, and then of course to the finite setting. The article is just providing a few thoughts in this direction and suggests to pay more attention to it, not just in the spirit of function spaces or pure functional analysis but more in the sense of constructive approximation theory, with quantitative error bounds, estimates for the required problem size if one needs a guaranteed estimate for the size of the error.

Thus in some sense the article describes the ideas and goals behind the material presented by the author during the conference in a more concrete but less reflected format. Important parts of those presentations are available in the form of PDF files from www.nuhag.eu.

Overall it is clear from this volume that harmonic analysis at large is and will provide a wide variety of interesting mathematical problems and that research in this direction will continue to be fruitful and rewarding for those interested in mathematical analysis in general, be it abstract or more application oriented.

Vienna, Austria Hans Feichtinger
October 2015

Preface

This book evolved from the written notes that were distributed to the students who participated in the CIMPA school, *New Trends in Applied Harmonic Analysis: Sparse Representations, Compressed Sensing and Multifractal Analysis*, which took place in Mar del Plata (Argentina) in August 2013.

This event was motivated by the recent interactions which developed between harmonic analysis and signal and image processing during the last 10 years. During that time, several technological deadlocks were solved through the resolution of deep theoretical problems in harmonic analysis. The purpose of this school was to focus on two particularly active areas which are representative of such advances: multifractal analysis and compressed sensing. The courses were taught by leaders in these areas and covered both theoretical aspects and applications. Most of the attendance was composed of PhD students and postdocs from diverse backgrounds (mathematics, signal and image processing, etc.), and the corresponding chapters of this book reflect the pedagogical care of the lecturers, in particular in the careful treatment of all needed prerequisites, and the illustration of the developments of each topic by several examples. Another original feature of this book is that some subjects overlap, with views taken from different perspectives, thus offering an in-depth picture of these scientific areas.

Let us be more specific. Multifractal analysis offers new tools of classification for signals and images derived from their scaling invariance properties. The part of the book concerning this subject include the contribution of K. Hare, "Multifractal Analysis of Cantor-like Measures," which deals with basics of fractal analysis and then focuses on the key example of Cantor-like measures. The contribution of Y. Heurteaux "An introduction to Mandelbrot cascades" goes one step further in modeling complexity and deals with the multifractal measures supplied by multiplicative cascades; a careful treatment of these examples is motivated both by the historical role played by these measures as models for the dissipation of energy in turbulent fluids and by the importance that they have recently acquired in other areas of mathematics (fragmentation, coalescence, harmonic measure associated with fractal sets, Schramm-Loewner evolution, etc.). Finally, the contribution of Stéphane Seuret "Multifractal analysis and Wavelets" deals with the extensions that these ideas have known in the setting of functions. The main tool here is wavelet analysis, a tool which is now prevalent in applied analysis and reappears in several other chapters of this book. Here its role is to yield a characterization of both pointwise and global regularity of functions. This property explains the success of wavelets in applied multifractal analysis, since this subject can be seen as unfolding the relationships between pointwise and global regularity and then deriving practical classification tools from these regularity characteristics.

Recently, many powerful techniques have been developed emphasizing the role of sparsity in signal and image processing. These new methods have had a substantial impact in areas like sampling, data compression and representation, atomic decompositions, wavelets, frames, and high-dimensional data analysis. In particular compressed sensing represents a new paradigm in signal and image processing, allowing to reconstruct compressible data from the knowledge of an underdetermined system, through an ℓ^1 minimization. The mathematics behind these methods is rich and sophisticated and presents new challenges. The chapters by Temlyakov "Lebesgue-type Inequalities for Greedy Approximation" and Hernández et. al "Results on Nonlinear Approximation for Wavelet Bases in Weighted Function Spaces" are excellent examples of the advances in this area.

On another note, just before the school took place, the *Kadison-Singer conjecture* was solved, and since this had deep impact on harmonic analysis – because of the implications with respect to the decomposition of frames into a finite number of Riesz bases *Feichtinger conjecture* – Pete Casazza gave a really nice lecture about the diverse attempts in the solution and agreed to write a chapter about all the implications.

Note that the contribution of Y. Heurteaux was not part of the courses taught at the CIMPA school of August 2013, but grew from the notes of another course taught at a fractal conference that took place in Porquerolles (France) in September 2013.

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- MinCyT, Ministerio de Ciencia y Tecnología, ARGENTINA
- IMU, International Mathematical Union

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Chapter 1
Multifractal Analysis of Cantor-Like Measures

Kathryn E. Hare

Abstract In this course we will study generalized Cantor sets and measures. We will see that they share many properties in common with self-similar sets and measures, although new geometric ideas are often needed in the proofs to replace the combinatorial structure of self-similar sets/measures. In particular, under a suitable separation condition the multifractal spectrum of generalized Cantor measures (the set of local dimensions) can be shown to be a closed interval, with one specific local dimension being attained at almost every point of the Cantor set.

Surprisingly, the property that the multifractal spectrum is a closed interval need not be true for convolutions of (even self-similar) Cantor measures. This seems to be a consequence of 'overlap' in their construction and was established first for certain examples of self-similar Cantor measures and subsequently for generalized Cantor measures. We will see that it is typically the case that the multifractal spectrum of a sufficiently large number of convolutions of fairly arbitrary, continuous measures admits an isolated point. This argument was motivated by the geometric ideas used in proving a special case of this property for generalized Cantor measures.

1.1 Introduction

Often in analysis one is interested in subsets of $\mathbb{R}$ of Lebesgue measure zero and the singular measures[1] concentrated on these sets. Many of the problems that arise have to do with quantifying the size of the set or the singularity of the measure; for such problems, fractal dimensions can be very helpful.

[1] By a measure, we mean a finite, positive, regular, compactly supported, Borel measure on $\mathbb{R}$.

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The classical middle-third Cantor set and its associated uniform measure is an important example of such a set and measure. The Cantor set and measure are often introduced in real analysis courses to illustrate unusual ideas or pathological behaviour. In this course, we will discuss generalizations of the classical Cantor set and measure, and investigate fractal concepts that help to quantify their singularity, such as local dimension and multifractal spectrum. These generalizations have interesting and unusual properties.

Generalized Cantor sets and measures are typically not self-similar and thus need not have the same symmetry or uniformity as the classical Cantor set/measure. Consequently, the concentration of the measure can vary at different points in its support, meaning general Cantor measures typically take on a range of different local dimensions. These different values are known as the multifractal spectrum. The study of the multifractal spectrum and the 'size' of the sets on which a given local dimension is attained is known as multifractal analysis.

For self-similar measures arising from an IFS which satisfies the open set condition, it is well known that the multifractal spectrum is a closed interval and formulas have been established for the Hausdorff dimension of the sets on which a given local dimension occurs. We will modify this argument to show that a similar result can be obtained for generalized Cantor measures, under reasonably weak assumptions. Another interesting fact we will establish is that the 'average' value of the local dimensions is attained at almost every point. These results can be found in Section 1.3.

Convolutions of the classical Cantor measure are again self-similar measures. However, they are not necessarily generated by an IFS that satisfies the open set condition so the general multifractal theory does not apply. In fact, the theory can fail in a striking way: the multifractal spectrum of the 3-fold convolution of the classical Cantor measure contains an isolated point. Here we will see that convolutions of quite general, continuous, probability measures typically admit isolated points in their multifractal spectrum, provided the number of convolutions is sufficiently large. In particular, this is the case for many generalized Cantor measures. These ideas are the content of Section 1.4.

Most of the proofs given in this note can be found in the literature, as detailed in the final section. There are many other important research papers on related topics; we have only mentioned those most relevant for the material discussed in the course.

1.2 Notation and Basic Facts

1.2.1 The Classical Cantor Set and Measure

The *classical middle-third Cantor set* C is a fascinating set which is often used in analysis to construct interesting examples. It is compact, totally disconnected, perfect (meaning, every point is an accumulation point), uncountable and of Lebesgue measure zero. By the *classical Cantor measure* we mean the singular, probability measure on $\mathbb{R}$ that is uniformly distributed on C. This measure, μ, can be defined in several equivalent ways:

1. As the self-similar measure that arises from the iterated function system (IFS) with contractions $F_i(x) = x/3 + 2i/3$, $i = 0,1$ and probabilities $1/2, 1/2$. This means the measure is invariant in the sense that
$$\mu(E) = \frac{1}{2}\left(\mu \circ F_0^{-1}(E) + \mu \circ F_1^{-1}(E)\right) \text{ for all Borel sets } E.$$
The classical Cantor set C is the self-similar set associated with this IFS.
2. As the Borel measure supported on C that assigns mass 2^{-k} to the Cantor intervals that arise at step k in the construction of the Cantor set.
3. As the weak limit of the discrete probability measures $\mu_K = 2^{-K}\sum_{j=1}^{2^K} \delta_{x_j}$, where $x_1, \ldots, x_{2^K}$ are the left end points of the 2^K Cantor intervals that are constructed at step K in the standard Cantor set construction. By a weak limit, we mean that for all continuous functions f on $[0,1]$ it is the case that $\int_0^1 f d\mu = \lim_K \int_0^1 f d\mu_K$.
4. As the probability measure whose cumulative distribution function is the Cantor ternary function.

From these different (but equivalent) descriptions of the Cantor measure one can easily establish many properties of the Cantor set/measure. Definition (2), for example, is useful in calculating the Hausdorff dimension of the set. From definition (3) it can be seen that the Fourier transform of μ is given by $\widehat{\mu}(y) = \prod_{k=1}^{\infty}(1 + e^{-4\pi i 3^{-k} y})/2$ for all y. Since the Cantor ternary function is a continuous function, it follows immediately from definition (4) that the Cantor measure is a continuous (or non-atomic) measure, meaning the measure of any singleton is 0.

The classical Cantor set and measure has been generalized in many ways. One obvious generalization is to consider the self-similar set arising from the IFS with contractions $F_i(x) = rx + i(1-r)$, $i = 0,1$ where $0 < r < 1/2$. This is the Cantor set with ratio of dissection r (rather than $1/3$rd, as in the classical case), meaning that at each step in the standard Cantor set construction one keeps the two outer closed intervals whose length is r times that of the parent interval. We will denote this Cantor set as $C(r)$, so that with this notation the classical Cantor set is $C(1/3)$. We can again define the associated *uniform Cantor measure* that assigns mass 2^{-k} to the Cantor intervals at step k, which in this case are of length r^k. This is the self-similar measure generated by the IFS given above, with probabilities $1/2, 1/2$.

Alternatively, rather than the uniform Cantor measure, we could consider the self-similar measure generated by the same iterated function systems again, but with probabilities p and $1-p$, where $0 \le p \le 1$. We call this the *p-Cantor measure* on $C(r)$. If $p = 0$ or 1, the p-Cantor measure is the point mass measure at 0 or 1, respectively. In all other cases, it is a continuous, singular, probability measure.

1.2.2 Cantor Sets and Measures with Varying Ratios of Dissection

In fractal geometry one is often interested in studying self-similar sets and measures arising from quite general iterated function systems. The IFS structure makes it possible to compute many important quantities and deduce various properties of the sets and measures. At the same time, the structure limits the kinds of examples that arise. If we relax this structure, we can create many other intriguing examples. One such variation is to allow the ratios of dissection in the construction of the Cantor set to vary at each step. We could also allow the probabilities to vary at different steps.

1.2.2.1 Cantor Sets with Varying Ratios of Dissection

Let $0 < r_j < 1/2$. We denote by $C(r_j)$ [2] the Cantor set with varying ratios of dissection, r_j at step j, given by the following iterative Cantor-like construction: Let $C_0 = [0,1]$. Remove from C_0 the open middle interval of length $1-2r_1$, leaving two closed intervals of lengths r_1. Call these intervals the *Cantor intervals* of step one and their union C_1. At step j in the construction assume we have inductively constructed C_j as a union of 2^j closed intervals of length $r_1 \cdots r_j$, the Cantor intervals of step j. Remove the open middle interval of length $(1-2r_{j+1})r_1 \cdots r_j$ from each of the step j intervals and let C_{j+1} be the union of the remaining 2^{j+1} closed intervals of length $r_1 \cdots r_{j+1}$. Finally, define the Cantor set $C(r_j)$ by

$$C(r_j) = \bigcap_{j=1}^{\infty} C_j.$$

As with the classical Cantor set, $C(r_j)$ is compact, perfect, totally disconnected and uncountable. Its Lebesgue measure is $\liminf_{n\to\infty} 2^{-n} r_1 \cdots r_n$ and hence is zero if, for instance, the r_j are bounded away from $1/2$.

1.2.2.2 Labelling Cantor Intervals and the Elements of the Cantor Set

The Cantor intervals from this construction can be labelled by finite words with letters from $\{0,1\}$. The Cantor intervals of step one will be denoted I_0 (left interval) and I_1 (right interval). In general, if the Cantor interval of step n is labelled by the word w of length n, then its two descendants are I_{w0} and I_{w1}. Each $x \in C(r_j)$ belongs to a unique Cantor interval of step n for each n and these intervals are descendants of one another. Thus x corresponds to an infinite word w with the property that if $w|n$ denotes the truncation of w to length n, then $I_{w|n}$ is the step n Cantor interval to which x belongs. When we write $x = (w_j)$ we mean this correspondence.

[2] More properly, we should write $C(\{r_j\})$, but we prefer $C(r_j)$ for simplicity. This should not cause any confusion with the notation $C(r)$ for the Cantor set with fixed ratio of dissection r.

1.2.2.3 Uniform and p-Cantor Measures

Given $0 \leq p \leq 1$, by the *p-Cantor measure* associated with $C(r_j)$, we mean the probability measure μ with the property that

$$\mu(I_{w0}) = \mu(I_w)p \text{ and } \mu(I_{w1}) = \mu(I_w)(1-p).$$

Thus if $w = (w_1, \ldots, w_n)$ with $w_i \in \{0,1\}$, then $\mu(I_{w_1 \cdots w_n}) = p^{n_0}(1-p)^{n-n_0}$ where $n_0 = card\{i : w_i = 0\}$. As in the case for Cantor sets with fixed ratio of dissection, the p-Cantor measure μ is a singular measure whose support is the Cantor set $C(r_j)$. It is continuous provided $p \neq 0, 1$. If $p = 1/2$, we call μ the *uniform Cantor measure* on $C(r_j)$.

More generally, given a sequence of weights $\{p_j\}$, $0 \leq p_j \leq 1$, we could define a Cantor measure by the rule $\mu(I_{w_1 \ldots w_n}) = p_{w_1 1} p_{w_2 2} \cdots p_{w_n n}$ where $p_{0j} = p_j$ and $p_{1j} = 1 - p_j$.

One could consider still more general Cantor sets and measures by removing from $[0,1]$, k_1 equally spaced, open intervals of length g_1 at step one, so that C_1 is the union of $k_1 + 1$ closed intervals of length r_1 where $(k_1+1)r_1 + k_1 g_1 = 1$. Then inductively remove from each Cantor interval of step j, k_j equally spaced open intervals of length g_j so that C_j is the union of $\prod_{i=1}^{j}(k_j + 1)$ closed intervals of length $r_1 \cdots r_j$ where $(k_j+1)r_j + k_j g_j = 1$. We can also define a general Cantor measure by putting weights p_{ij} on the $i = 1, \ldots, k_j + 1$ descendants at step j. In this note, we will focus on p-Cantor measures on $C(r_j)$, but much of what is said here is true for these very general Cantor sets and measures, at least under suitable assumptions. The technical details will be left for the reader.

1.2.3 Hausdorff Dimension

Let $\delta > 0$. By a δ-cover of a non-empty Borel subset $E \subseteq \mathbb{R}$ we mean a countable collection of sets $\{U_i\}$ of diameter at most δ, whose union contains E. We write $|U_i|$ to denote the diameter of the set U_i. Given $s \geq 0$, we define

$$H^s_\delta(E) = \inf\left\{ \sum_{i=1}^{\infty} |U_i|^s : \{U_i\} \text{ is a } \delta\text{-cover of } E \right\}$$

and put

$$H^s(E) = \sup_{\delta>0} H^s_\delta(E) = \lim_{\delta \to 0^+} H^s_\delta(E).$$

$H^s(\cdot)$ is a measure known as the *s-dimensional Hausdorff measure*. $H^s(E)$ is a decreasing function of s and can be positive and finite for at most one choice of s. The *Hausdorff dimension* of E, denoted $\dim_H E$, is defined to be the unique index s such that $H^t(E) = 0$ if $t > s$ and $H^t(E) = \infty$ for $t < s$. Thus

$$\begin{aligned}\dim_H F &= \inf\{s : H^s(F) = 0\} \\ &= \sup\{s : H^s(F) = \infty\}.\end{aligned}$$

A useful fact is the Mass distribution principle: If there are a measure μ on E and real numbers $c, \delta > 0$ such that $\mu(U) \leq c|U|^s$ for all Borel sets U with diameter at most δ, then $H^s(E) \geq \mu(E)/c$ and $\dim_H E \geq s$.

We leave it as an exercise to verify that the Hausdorff dimension of $C = C(r_j)$ is given by the formula

$$\dim_H C = \liminf_{n\to\infty} \frac{\log 2}{\frac{1}{n}|\log r_1 \cdots r_n|}.$$

Exercise 1.1. Establish the formula given for the Hausdorff dimension of $C(r_j)$.

Exercise 1.2. Show that for every $s \leq 1$ there is a Cantor set with Hausdorff dimension equal to s.

Exercise 1.3. Construct a Cantor-like set, $C(r_j)$, with Hausdorff dimension one and Lebesgue measure zero.

1.3 Multifractal Analysis of p-Cantor Measures

1.3.1 Local Dimension

In many problems one is interested in quantifying the singularity of a measure, i.e., to specify, in some sense, how concentrated the measure is. One way to quantify this is through the *Hausdorff dimension of the measure* μ. This is defined as

$$\dim_H \mu = \inf\{\dim_H E : \mu(E) > 0\}.$$

This quantity provides global information on the singularity of the measure μ. For measures that are not uniformly distributed it is also of interest to quantify their local singularity. The local dimension is useful for this.

Definition 1.1. By the local dimension at x of a probability measure μ on $\mathbb{R}$ we mean the quantity

$$dim_{loc}\mu(x) = \lim_{r\to 0^+} \frac{\log(\mu(B(x,r)))}{\log r}$$

where $B(x,r)$ is the ball centred at x with radius r, provided this limit exists.

The upper and lower dimensions, denoted $\overline{dim}_{loc}\mu(x)$ and $\underline{dim}_{loc}\mu(x)$, are obtained by replacing the limit in the definition above with lim sup and lim inf, respectively.

The local dimension at x describes the power law behaviour of $\mu(B(x,r))$ for small r. Notice that if $x \notin \operatorname{supp}\mu$, then $dim_{loc}\mu(x) = \infty$, while if μ is Lebesgue measure on $[0,1]$, $dim_{loc}\mu(x) = 1$ at all $x \in [0,1]$.

One can prove that

$$\dim_H \mu = \sup\{s : \underline{dim}_{loc}\mu(x) \geq s \text{ for } \mu \text{ a.e. } x\}.$$

Moreover, the following is true.

Proposition 1.1. *Suppose μ is a probability measure, $F \subseteq \mathbb{R}$ is a Borel set and $0 < c < \infty$.*

(a) $H^s(F) \geq \mu(F)/c$ if

$$\limsup_{r \to 0^+} \frac{\mu(B(x,r))}{r^s} \leq c \text{ for all } x \in F.$$

(b) $H^s(F) \leq 10^s \mu(\mathbb{R})/c$ if

$$\limsup_{r \to 0^+} \frac{\mu(B(x,r))}{r^s} \geq c \text{ for all } x \in F.$$

Proof. (a) Fix $\varepsilon > 0$ and for each n let

$$F_n = \{x \in F : \mu(B(x,r)) \leq (c+\varepsilon)r^s \text{ for all } r \leq 1/n\}.$$

The sets F_n are increasing and the assumption of (a) guarantees that their union is all of F.

Temporarily fix n and let $\{U_i\}$ be a $1/2n$-cover of F and hence also of F_n. Each set U_i has diameter less than $1/n$ and thus $\mu(B(x,|U_i|)) \leq (c+\varepsilon)|U_i|^s$ for all $x \in F_n$. Notice that if $x \in U_i \cap F_n$, then $B(x,|U_i|) \supseteq U_i$ and $\mu(U_i) \leq (c+\varepsilon)|U_i|^s$. Thus

$$\mu(F_n) \leq \sum_{i:U_i \cap F_n \neq \text{empty}} \mu(U_i) \leq (c+\varepsilon)\sum |U_i|^s.$$

This is true for all $1/2n$-covers of F and consequently $\mu(F_n) \leq (c+\varepsilon)H^s_{1/2n}(F)$. But as $n \to \infty$, $\mu(F_n) \to \mu(F)$ and $H^s_{1/2n}(F) \to H^s(F)$. Since $\varepsilon > 0$ was arbitrary, $\mu(F) \leq cH^s(F)$.

(b) Fix $\varepsilon, \delta > 0$ and consider the collection of all balls, $B(x,r)$, with $x \in F$, $0 < r < \delta$ and $\mu(B(x,r)) \geq (c-\varepsilon)r^s$. By assumption, every $x \in F$ belongs to such a ball for arbitrarily small r. By the Vitali covering lemma there are countably many disjoint balls from this collection, $\{B_i\}$, such that $\mu(F \setminus \bigcup_i B_i) = 0$ and every ball in the collection is contained in the union of the sets $\widetilde{B}_i$, where $\widetilde{B}_i$ is a ball concentric with B_i and having five times the radius. Thus $F \subseteq \bigcup_i \widetilde{B}_i$ and $\left|\widetilde{B}_i\right|^s \leq 10^s \mu(B_i)/(c-\varepsilon)$. As $\left|\widetilde{B}_i\right| \leq 10\delta$ and the sets B_i are disjoint,